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THE PROPAGATION OF THE ENERGY OF ELASTIC WAVES IN ANISOTROPIC MEDIA†

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The particular features of the propagation of the seismic energy of elastic waves in anisotropic media with four constants of elasticity, depending on the directions of motion of the waves and the ratios of the constants of elasticity for all practical media of the anisotropy class considered, are investigated. A direct connection is established between the formation of acute-angled edges on the fronts of quasi-transverse waves from point sources and the distinctive features of the propagation of the energy of the waves under certain conditions for the constants of elasticity. © 2003 Elsevier Ltd. All rights reserved.

A study of the particular features of the propagation of the energy of plane elastic waves in anisotropic media is of particular interest, since the vectors of the energy flux density, from the physical point of view, determine the directions of propagation of the wave fronts, and almost all seismic fields can be considered approximately as locally plane waves. Some problems of the propagation of seismic energy in anisotropic media were considered previously in [1–5], but the dependence of the propagation of the energy fluxes on the directions of motion of the waves and on the ratios of the constants of elasticity of the media were not investigated.

1. PLANE WAVES IN ANISOTROPIC MEDIA

We will consider an anisotropic medium with four constants of elasticity. The x, y, z axes of a rectangular system of coordinates coincide with the axes of elastic symmetry of the medium, and the oscillations are independent of the z coordinate.

The equations of motion in terms of displacements have the form [6]

$$au_{xx} + du_{yy} + cv_{xy} = u_t, \quad cu_{xy} + dv_{xx} + bv_{yy} = v_t \tag{1.1}$$

The ratios of the constants of elasticity to the density of the medium

$$a = C_{11}/\rho, \quad b = C_{22}/\rho, \quad d = C_{66}/\rho, \quad c = (C_{66} + C_{12})/\rho$$

satisfy the necessary and sufficient conditions for the form of the elastic energy to be positive-definite

$$a > d, \quad b > d, \quad d > 0, \quad K_1 = ab - (c-d)^2 > 0 \tag{1.2}$$

The solutions of Eqs (1.1), expressing plane waves, have the form [6]

$$\begin{gathered} u_k = (r_k - c\theta\lambda_k)w_k(\Omega_k), \quad v_k = -(p_k - c\theta\lambda_k)w_k(\Omega_k) \\ p_k = a\theta^2 + d\lambda_k^2 - 1, \quad r_k = d\theta^2 + b\lambda_k^2 - 1, \quad p_k r_k = c^2\theta^2\lambda_k^2 \\ \Omega_k = t - \theta x + \lambda_k y, \quad k = 1, 2 \end{gathered} \tag{1.3}$$

where

$$\begin{gathered} \lambda_k = [A + (-1)^k(A^2 - B)^{1/2}]^{1/2}/(2bd)^{1/2} \\ A = (b+d) - L\theta^2, \quad B = 4abd^2(1/a - \theta^2)(1/d - \theta^2), \quad L = ab + d^2 - c^2 \end{gathered} \tag{1.4}$$

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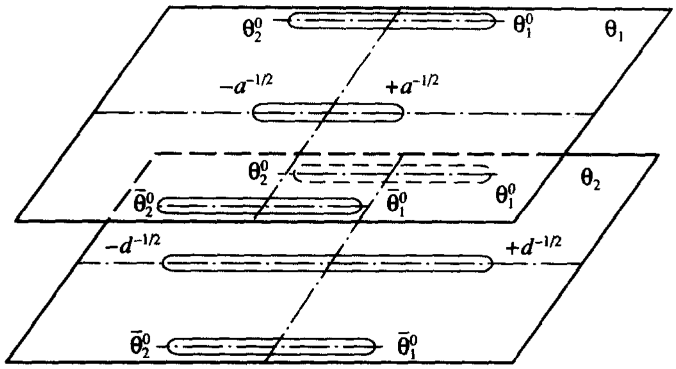

Fig. 1

The function λ_1 and λ_2 are branches of the algebraic function λ, uniquely defined on the Riemann surface. The functions w_1 and w_2 are arbitrary continuous twice-differentiable functions, if the coefficients in them with variable values are real; if some of these coefficients in some region of space x, y, t are complex quantities, w_k will be regarded as analytical functions in this region.

The inner radicals of functions (1.4) have branching points [7, 8]

$$\theta_i^0 = \pm\{[M \pm (-4bdc^2N_1)^{1/2}]/(K_1K_2)\}^{1/2}$$
$$N_1 = (a-d)(b-d)-c^2, \quad K_1 = ab-(c-d)^2, \quad K_2 = ab-(c+d)^2 \qquad (1.5)$$
$$M = bN_1 + dN_7, \quad N_7 = (b-d)^2 - c^2$$

which may be complex, imaginary or real depending on the ratios of the constants of elasticity.

When the condition

$$N_2 = (a-d)b - c^2 > 0 \qquad (1.6)$$

is satisfied [7, 8], the branching points for the outer radicals of (1.4) are the points $\theta_1 = \pm a^{-1/2}$ when $k = 1$ and the points $\theta_2 = \pm d^{-1/2}$ when $k = 2$. In this case the Riemann surface consists of planes θ_1 and θ_2 with cuts $(-a^{-1/2}, +a^{-1/2})$ and $(-d^{-1/2}, +d^{-1/2})$, joined crosswise along the cuts connecting the branching points θ_i^0. In Fig. 1 we show the Riemann surface for the case when the branching points θ_i^0 are pair wise complex conjugate.

On the edges of the cuts $(-a^{-1/2}, +a^{-1/2})$ of the θ_1 plane and $(-d^{-1/2}, +d^{-1/2})$ of the θ_2 plane the functions λ_2 and λ_2 have real values, and the functions (1.3) express real plane waves: quasi-longitudinal when $k = 1$ and quasi-transverse when $k = 2$.

When $N_2 < 0$ the outer radical of the function λ_1 has four branching points $\theta_1 = \pm a^{-1/2}$ and $\theta_2 = \pm d^{-1/2}$, but the outer radical of the function λ_2 has no branching points. Of the four branching points for the inner radical of the functions λ_1 and λ_2 we have the following: two real $\pm\theta_1^0$ and two imaginary $\pm\theta_2^0$, where $\theta_1^0 > d^{-1/2}$. The function λ_1 is single valued in the θ_1 plane with cuts $(-a^{-1/2}, +a^{-1/2})$, $(\pm d^{-1/2}, \pm\theta_1^0)$ and $(\pm\theta_1^0, \pm\infty)$ along the real axis and with cuts $(\pm\theta_2^0, \pm i\infty)$ along the imaginary axis. The function λ_2 is single valued in the θ_2 is single valued in the θ_2 plane with cuts $(-\theta_1^0, +\theta_1^0)$ and $(\pm\theta_1^0, \pm\infty)$ along the real axis and with cuts $(\pm\theta_2^0, \pm i\infty)$ along the imaginary axis. The Riemann surface (Fig. 2) consists of the θ_1 and θ_2 planes, joined crosswise along the edges of the cuts $(\pm\theta_1^0, \pm\infty)$ and $(\pm\theta_2^0, \pm i\infty)$. On the edges of the cuts $(-a^{-1/2}, +a^{-1/2})$, $(\pm d^{-1/2}, \pm\theta_1^0)$ of the θ_1 plane and $(-\theta_1^0, +\theta_1^0)$ of the θ_2 plane the functions λ_1 and λ_2 have real values. The functions (1.3) express real plane waves [6]: quasi-longitudinal when $k = 1$ on the edges of the cut $(-a^{-1/2}, +a^{-1/2})$ of the θ_1 plane, quasi-transverse when $k = 2$ on the edges of the cut $(-\theta_1^0, +\theta_1^0)$ of the θ_2 plane and when $k = 1$ on the edges of the cuts $(\pm d^{-1/2}, \pm\theta_1^0)$ of the θ_1 plane.

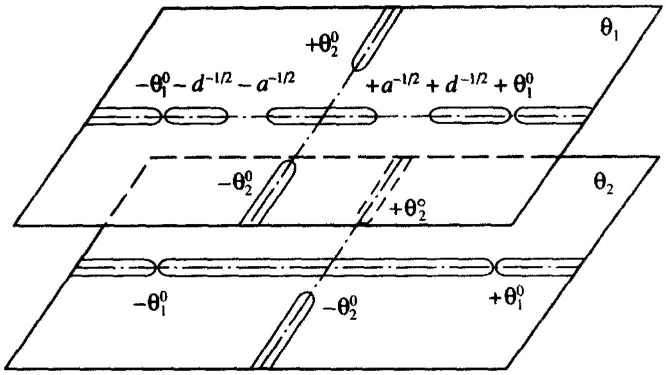

Fig. 2

2. THE ENERGY FLUXES AND THE RAY VELOCITY

The propagation of the elastic waves is related to the propagation of energy in the deformed medium.

The projections of the energy flux density vector onto the coordinate axes have the form [9]

$$S_x = -[u_t\sigma_x + v_t\tau_{xy}], \quad S_y = -[u_t\tau_{xy} + v_t\sigma_y] \tag{2.1}$$

where u_t and v_t are the derivatives of the components of the displacements with respect to time.

The components of the stresses for the case considered can be expressed in terms of the derivatives of the components of the displacements by the formulae

$$\sigma_x = \rho[au_x + (c-d)v_y], \quad \sigma_y = \rho[(c-d)u_x + bv_y], \quad \tau_{xy} = \rho d[u_y + v_x] \tag{2.2}$$

From expressions (2.1) and (2.2) we have

$$\begin{aligned} S_x &= -\rho\{u_t[au_x + (c-d)v_y] + v_t[d(u_y + v_x)]\} \\ S_y &= -\rho\{u_t[d(u_y + v_x)] + v_t[(c-d)u_x + bv_y]\} \end{aligned} \tag{2.3}$$

Substituting the values of the derivatives of the functions (1.3) into formulae (2.3), we obtain the following expressions for the projections onto the coordinate axes of the energy flux density vectors of the quasi-longitudinal and quasi-transverse waves

$$\begin{aligned} S_{xk} &= \rho\theta p_k^{-1}(p_k - c\theta\lambda_k)^2 Q_k[w_k^1(\Omega_k)]^2 \\ S_{yk} &= -\rho\lambda_k p_k^{-1}(p_k - c\theta\lambda_k)^2 M_k[w_k^1(\Omega_k)]^2 \end{aligned} \tag{2.4}$$

where

$$Q_k = 2ad\theta^2 + L\lambda_k^2 - (a+d), \quad M_k = 2bd\lambda_k^2 + L\theta^2 - (b+d) \tag{2.5}$$

The phase velocities of the waves (1.3), which determine the propagation of the wave fronts in the directions of the normals, are expressed by the formulae [7, 10]

$$b_k = (\theta^2 + \lambda_k^2)^{-1/2} \tag{2.6}$$

The ray velocities of the waves (1.3), which define the propagation of the wave fronts in the directions of the energy flux density vectors, are related to the phase velocities by the relations [3]

$$b_k = (n_k \cdot c_k) \tag{2.7}$$

where c_k are the ray velocity vectors. By relations (2.6) and (2.7) the ray velocities are defined by the formulae

$$c_k = [(\theta^2 + \lambda_k^2)^{1/2}\cos\varphi_k]^{-1} \tag{2.8}$$

where φ_k are the angles formed by the vectors of the ray velocities with the vectors of the phase velocities.

For any directions of propagation of the waves (1.3), the ray and phase velocities satisfy the conditions $c_k \geqslant b_k$.

We will denote the angles formed by the vectors of the phase velocities, the ray velocities and the displacements of the particles of the medium with the negative semi-axis y by α_k, β_k, γ_k, which, by relations (1.3) and (2.4), are given by the formulae

$$\operatorname{tg}\alpha_k = \theta\lambda_k^{-1}, \quad \operatorname{tg}\beta_k = \theta Q_k(\lambda_k M_k)^{-1}, \quad \operatorname{tg}\gamma_k = (r_k - c\theta\lambda_k)(p_k - c\theta\lambda_k)^{-1} \tag{2.9}$$

The angles φ_k, formed by the vectors of the ray velocities with the vectors of the phase velocities, taking into account that $\varphi_k = \beta_k - \alpha_k$, are given by the formulae

$$\operatorname{tg}\varphi_k = \theta\lambda_k(Q_k - M_k)(\theta^2 Q_k + \lambda_k^2 M_k)^{-1} \tag{2.10}$$

(the angles are measured from the normals to the wave fronts in an anticlockwise direction).

The features of the propagation of the energy of the elastic waves are determined by the quantities N_1, N_2, K_2 and M, and also by the quantities

$$N_3 = (b-d)a - c^2, \quad N_4 = a - d - c, \quad N_5 = b - d - c, \quad N_6 = (a-d)^2 - c^2,$$
$$N_7 = (b-d)^2 - c^2 \tag{2.11}$$

3. ANALYSIS OF THE SOLUTIONS WHEN $N_2 > 0$ AND $N_3 > 0$

When $N_2 > 0$ the solutions obtained are uniquely defined on the Riemann surface shown in Fig. 1. Since the x and y axes coincide with the axes of elastic symmetry of the medium, the wave processes considered can be investigated sufficiently for values of θ in the ranges

$$0 \le \theta \le a^{-1/2}, \quad 0 \le \theta \le d^{-1/2} \tag{3.1}$$

of the upper edges o the cuts of the θ_1 and θ_2 planes (Fig. 1).

On the boundaries of the ranges (3.1) the components of the energy flux density vectors (2.4) of the quasi-longitudinal waves ($k = 1$) and quasi-transverse waves ($k = 2$) take the values

$$\begin{aligned}
&S_{x1}(0) = 0, \quad S_{y1}(0) = -\rho(b-d)^2 b^{-3/2}[w_1^1(\Omega_1^0)]^2 \\
&S_{x1}(a^{-1/2}) = \rho(a-d)^2 a^{-3/2}[w_1^1(\Omega_1^*)]^2, \quad S_{y1}(a^{-1/2}) = 0 \\
&S_{x2}(0) = 0, \quad S_{y2}(0) = -\rho(b-d)^2 d^{-3/2}[w_2^1(\Omega_2^0)]^2 \\
&S_{x2}(d^{-1/2}) = \rho(a-d)^2 d^{-3/2}[w_2^1(\Omega_2^*)]^2, \quad S_{y2}(d^{-1/2}) = 0 \\
&\Omega_1^0 = t + b^{-1/2}y, \quad \Omega_1^* = t - a^{-1/2}x, \quad \Omega_2^0 = t + d^{-1/2}y, \quad \Omega_2^* = t - d^{-1/2}x
\end{aligned} \tag{3.2}$$

Consequently, the energy flux density vectors of the quasi-longitudinal waves ($k = 1$) and quasi-transverse waves ($k = 2$) when $\theta = 0$ are directed along the negative y semi-axis, and when $\theta = a^{-1/2}$ and $\theta = d^{-1/2}$ they are directed along the positive x semi-axis.

The functions λ_1 and λ_2 in the corresponding ranges (3.1) decrease continuously in the intervals [8]

$$b^{-1/2} \ge \lambda_1 \ge 0, \quad d^{-1/2} \ge \lambda_2 \ge 0 \tag{3.3}$$

The function p_k and r_k ($k = 1, 2$) in the corresponding ranges (3.1) satisfy the conditions

$$p_1 < 0, \quad r_1 < 0, \quad p_2 > 0, \quad r_2 > 0 \tag{3.4}$$

At the boundaries of the regions (3.1) the function Q_k and M_k take the values

$$Q_1(0) = -R_2b^{-1},\quad Q_1(a^{-1/2}) = -(a-d),\quad M_1(0) = -(b-d)$$
$$M_1(a^{-1/2}) = -R_1a^{-1},\quad Q_2(0) = N_3d^{-1},\quad Q_2(d^{-1/2}) = (a-d) \tag{3.5}$$
$$M_2(0) = (b-d),\quad M_2(d^{-1/2}) = N_2d^{-1},\quad R_1 = (a-d)d+c^2,\quad R_2 = (b-d)d+c^2$$

The derivatives of the functions Q_k and M_k have the form

$$Q_{k\theta} = \theta A_k(bd)^{-1},\quad M_{k0} = 2\theta B_k,\quad D = [K_1K_2\theta^4 - 2M\theta^2 + (b-d)^2]^{1/2}$$
$$A_k = -K_1K_2 + (-1)^k L(K_1K_2\theta^2 - M)D^{-1},\quad B_k = (-1)^k(K_1K_2\theta^2 - M)D^{-1} \tag{3.6}$$

The function D, apart from a positive constant factor, is the inner radical in expression (1.4), which is positive in the intervals (3.1).

On the boundaries of the intervals (3.1) the functions A_k and B_k take the values

$$A_1(0) = -F_3,\quad A_1(a^{-1/2}) = -a(b+d)R_1^{-1}F_4,\quad A_2(0) = -F_5$$
$$A_2(d^{-1/2}) = d(b+d)N_2^{-1}F_4,\quad B_1(0) = (b-d)^{-1}M \tag{3.7}$$
$$B_1(a^{-1/2}) = -R_1^{-1}F_1,\quad B_2(0) = -(b-d)^{-1}M,\quad B_2(d^{-1/2}) = N_2F_2,$$

where

$$F_1 = K_1K_2 - aM,\quad F_2 = K_1K_2 - dM,\quad F_3 = K_1K_2 - (b-d)^{-1}LM$$
$$F_4 = K_1K_2 - (b+d)^{-1}LM,\quad F_5 = K_1K_2 + (b-d)^{-1}LM \tag{3.8}$$

It can be shown that when $N_2 > 0$ the coefficients of M in expressions (3.8) satisfy the conditions

$$a > (b-d)^{-1}L > (b+d)^{-1}L > d \tag{3.9}$$

if

$$R_3 = (c^2-d^2) - ad > 0 \tag{3.10}$$

Where $R_3 < 0$ under conditions (3.9) we have $(b-d)^{-1}L > a$.

Expressions (3.8) reduce to the form

$$F_1 = bdN_6 - (c^2-d^2)N_1,\quad F_2 = [(ab-c^2) + (a-d)d]N_1 + d^2N_6$$
$$F_3 = 2d(b-d)^{-1}[(c^2-d^2)N_1 - adN_7],\quad F_4 = 2bd[aN_1 + dN_6](b+d)^{-1} \tag{3.11}$$
$$F_5 = 2b(b-d)^{-1}\{[(ab-c^2) + (b-d)d]N_1 + d^2N_7\}$$

The derivatives of the functions A_k and B_k have the same values, apart from a constant factor $L > 0$,

$$A_{k\theta} = LB_{k\theta} = (-1)^k 8bdc^2LN_1\theta D^{-3/2} \tag{3.12}$$

When $N_2 > 0$ and $N_3 > 0$, the quantity N_1 may have different signs.

Case 1. When $N_1 > 0$, the derivatives (3.12) satisfy the conditions

$$A_{i\theta} < 0,\quad B_{i\theta} < 0,\quad A_{2\theta} > 0,\quad B_{2\theta} > 0 \tag{3.13}$$

Taking conditions (1.2) into account, from the equation

$$(b-d)^2K_1K_2 - M^2 = 4bdc^2N_1 \tag{3.14}$$

we conclude that $K_2 > 0$. When $N_1 > 0$, the quantities N_4 and N_5 may have the same (positive) signs or opposite signs.

Case 1(a). If $N_4 > 0$ and $N_5 > 0$, we have $N_6 > 0$ and $N_7 > 0$, and according to expressions (3.11) $F_i > 0$ when $i = 2, 4$ and 5, while F_1 and F_3 may take positive and negative values. It follows from (1.5) that $M > 0$.

When $M > 0$, $K_2 > 0$, $R_3 > 0$, according to the conditions (3.9), the following combinations of the distribution of the values of F_i are possible

$$\begin{gathered} F_1 < 0, \quad F_5 > F_2 > F_4 > F_3 > 0 \\ F_5 > F_2 > F_4 > F_3 > F_1 > 0 \\ F_1 < F_3 < 0, \quad F_5 > F_2 > F_4 > 0 \end{gathered} \tag{3.15}$$

When the first combination of conditions (3.15) is satisfied, by expressions (3.7) we have

$$\begin{gathered} A_1(0) < 0, \quad A_1(a^{-1/2}) < 0, \quad B_1(0) > 0, \quad B_1(a^{-1/2}) > 0 \\ A_2(0) < 0, \quad A_2(d^{-1/2}) > 0, \quad B_2(0) < 0, \quad B_2(d^{-1/2}) > 0 \end{gathered} \tag{3.16}$$

It follows from relations (3.13) and (3.16) that in the intervals (3.1) the functions A_k and B_k satisfy the conditions

$$\begin{gathered} A_1(\theta) < 0, \quad B_1(\theta) > 0, \quad A_2(\theta) < 0 \ \ (\theta < \theta_{12}^*), \quad A_2(\theta) > 0 \ \ (\theta > \theta_{12}^*) \\ B_2(\theta) < 0 \ \ (\theta < \theta_2^*), \quad B_2(\theta) > 0 \ \ (\theta > \theta_2^*) \end{gathered} \tag{3.17}$$

where θ_{12}^* and θ_2^* are the zeros of the functions $A_2(\theta)$ and $B_2(\theta)$.

It follows from relations (3.6) and (3.17) that the derivatives of the functions Q_k and M_k in the intervals (3.1) satisfy the conditions

$$\begin{gathered} Q_{1\theta} < 0, \quad M_{1\theta} > 0, \quad Q_{2\theta} < 0 \ \ (\theta < \theta_{12}^*), \quad Q_{2\theta} > 0 \ \ (\theta > \theta_{12}^*) \\ M_{2\theta} < 0 \ \ (\theta < \theta_2^*), \quad M_{2\theta} > 0 \ \ (\theta > \theta_2^*) \end{gathered} \tag{3.18}$$

At the boundaries of the intervals (3.1), according to expressions (3.5), the functions Q_1 and M_1 have negative values while Q_2 and M_2 have positive values.

The functions Q_2 and M_2 have a minimum at the points θ_{12}^* and θ_2^* respectively. From the equations $A_2(\theta_{12}^*) = 0$ and $B_2(\theta_2^*) = 0$ the extreme points can be represented by the expressions

$$\theta_{12}^* = [M(K_1K_2)^{-1} + L^{-1}D_{12}^*]^{1/2}, \quad \theta_2^* = [M(K_1K_2)^{-1}]^{1/2} \tag{3.19}$$

where D_{12}^* is the value of the function D, defined by expression (3.6) for θ_{12}^*, when $D_{12}^* > 0$.

Since the function B_2 at the point $\theta = a^{-1/2}$ is less than zero, by expressions (3.19) we have the following distribution of the extreme points on the real θ axis

$$a^{-1/2} < \theta_2^* < \theta_{12}^* < d^{-1/2} \tag{3.20}$$

The minimum values of the functions Q_2 and M_2 satisfy the conditions

$$Q_2(\theta_{12}^*) = 2adL^{-1}D_{12}^* > 0, \quad M_2(\theta_2^*) = D_{12}^* > 0 \tag{3.21}$$

It follows from relations (3.5) and (3.18)–(3.21) that the functions Q_1 and M_1 in the first interval (3.1) have negative values, the function Q_1 decreases continuously, and the function M_1 increases continuously. In the first interval of (3.1), Q_2 and M_2 are positive functions, which take minimum values at the points θ_{12}^* and θ_2^*, satisfying condition (3.20).

Since at the boundaries of the intervals (3.1) the differences in the values (3.5) of the functions Q_k and M_k

$$\begin{gathered} |M_1(0) - |Q_1(0)| = b^{-1}N_7, \quad |Q_1(a^{-1/2})| - |M_1(a^{-1/2})| = a^{-1}N_6 \\ Q_2(0) - M_2(0) = d^{-1}N_1, \quad M_2(d^{-1/2}) - Q_2(d^{-1/2}) = d^{-1}N_1 \end{gathered} \tag{3.22}$$

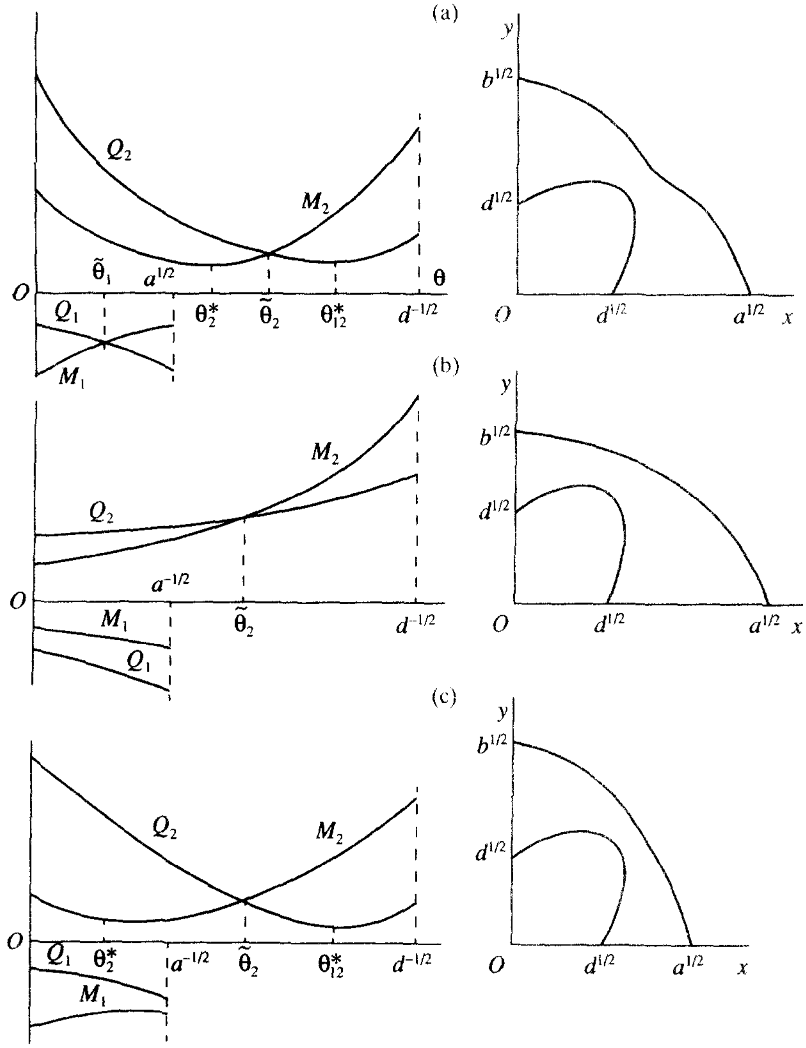

Fig. 3

are greater than zero, the graphs of the functions Q_k and M_k intersect and have the form shown in Fig. 3(a). The points of intersection are defined by the conditions $Q_k(\tilde{\theta}_k) = M_k(\tilde{\theta}_k)$ and have the coordinates

$$\tilde{\theta}_1 = [(b-d-c)K_2^{-1}]^{1/2}, \quad \tilde{\theta}_2 = [(b-d+c)K_1^{-1}]^{1/2} \tag{3.23}$$

The values of the functions Q_k and M_k at the points of intersection of the graphs are given by the expressions

$$Q_1(\tilde{\theta}_1) = M_1(\tilde{\theta}_1) = -c, \quad Q_2(\tilde{\theta}_2) = M_2(\tilde{\theta}_2) = c \tag{3.24}$$

It was established in [10] that when $N_4 > 0$ and $N_5 > 0$ the phase velocities of the quasi-longitudinal waves take minimum values at the point $\tilde{\theta}_1$, and the quasi-transverse waves take maximum values at the points $\tilde{\theta}_2$. Consequently, the points of intersection of the graphs of Q_k and M_k correspond to waves with extreme phase velocities.

If $F_1 > 0$, then according to relations (3.8) and (3.9) the second condition of (3.15) is satisfied. Repeating the discussion carried out for the first combination of conditions (3.15), we obtain the conditions for the derivatives of the functions Q_k and M_k, which differ from conditions (3.18) solely in the fact that the condition $M_{10} > 0$ is changed into

$$M_{1\theta} > 0 \ \ (\theta_1 < \theta_1^*), \quad M_{1\theta} < 0 \ \ (\theta > \theta_1^*) \tag{3.25}$$

The extreme points, defined by expressions (3.19), satisfy the condition

$$\theta_1^* = \theta_2^* < a^{-1/2} < \theta_{12}^* < d^{-1/2} \tag{3.26}$$

Hence, it also follows from expressions (3.5) and (3.22) that the graphs of the functions Q_k and M_k have the form shown in Fig. 3(a). The graph of the function M_1 differs in the fact that at the point θ_1^* it has a maximum $M_1(\theta_1^*) = -D_1^* < 0$; unlike distribution (3.20) the extreme points (3.19) satisfy condition (3.26).

If $F_3 < 0$, then by relations (3.8) and (3.9) the third condition of (3.15) is satisfied. The derivatives of the functions Q_k and M_k satisfy conditions which differ from conditions (3.18) in the fact that the first condition is replaced by

$$Q_{1\theta} > 0 \ (\theta < \theta_{11}^*), \quad Q_{1\theta} < 0 \ (\theta > \theta_{11}^*) \tag{3.27}$$

The extreme points (3.9) satisfy the condition

$$\theta_{11}^* < a^{-1/2} < \theta_2^* < \theta_{12}^* < d^{-1/2} \tag{3.28}$$

Hence it also follows from expressions (3.5) and (3.22) that the graphs of the functions Q_k and M_k have the form shown in Fig. 3(a), and the graph of the function Q_1 differs by having a maximum $Q_1(\theta_{11}^*) = -2adL^{-1}D_{11}^* < 0$ at the point θ_{11}^*.

Where $R_3 < 0$, conditions (3.9) and (3.15) are only slightly changed: in condition (3.9) we have $(b-d)^{-1}L > a$ and in conditions (3.15) $F_3 < F_1$. It can be shown that in these cases the graphs of the functions Q_k and M_k have a form similar to the graphs in Fig. 3(a).

Hence, when $N_4 > 0$ and $N_5 > 0$ the functions Q_k and M_k, defined in the intervals (3.1), satisfy the following conditions:

in the interval $(0, a^{-1/2})$

$$M_1 < Q_1 < 0 \ (\theta < \tilde{\theta}_1), \quad Q_1 < M_1 < 0 \ (\theta > \tilde{\theta}_1) \tag{3.29}$$

in the interval $(0, d^{-1/2})$

$$Q_2 > M_2 > 0 \ (\theta < \tilde{\theta}_2), \quad M_2 > Q_2 > 0 \ (\theta > \tilde{\theta}_2) \tag{3.30}$$

In the intervals (3.1) the angles α_1 and α_2, which determine the directions of the phase velocity vectors of the quasi-longitudinal ($k = 1$) and quasi-transverse ($k = 2$) waves (1.3), increase monotonically, according to relations (2.9) and (3.3).

When the waves (1.3) travel in the directions of the axes of elastic symmetry y and x of the medium, the directions of the phase and ray velocity vectors and of the displacements of the particles of the medium, according to relations (2.9) and (2.10), are determined by the following angles:

when $\theta = 0$

$$\alpha_k = \beta_k = \varphi_k = 0 \ (k = 1, 2), \quad \gamma_1 = 0, \quad \gamma_2 = \pi/2 \tag{3.31}$$

when $\theta = a^{-1/2}$ and $\theta = d^{-1/2}$

$$\alpha_k = \beta_k = \pi/2, \quad \varphi_k = 0, \quad \gamma_1 = \pi/2, \quad \gamma_2 = 0 \tag{3.32}$$

It follows from relations (3.31) and (3.32) that in this case the directions of the phase and ray velocity vectors of the quasi-longitudinal and quasi-transverse waves and of the displacement vectors of the particles of the medium of the quasi-longitudinal waves coincide with the directions of the normals to the wave fronts. The directions of the displacement vectors of the quasi-transverse waves coincide with the wave fronts. Consequently, in the directions of the axes of elastic symmetry the quasi-longitudinal and quasi-transverse waves become purely longitudinal and purely transverse waves.

For the waves (1.3) with extreme phase velocities (Fig. 3a) for values of $\theta = \tilde{\theta}_k$, defined by formulae (3.23) and of the corresponding points of intersection of the graphs of the functions Q_k and M_k (Fig. 3a), the directions of the phase and ray velocity vectors and of the displacements of the particles of the medium are determined by the following angles

$$\tilde{\alpha}_1 = \tilde{\beta}_1 = \tilde{\gamma}_1 = \operatorname{arctg}(N_5 N_4^{-1})^{1/2}, \quad \tilde{\varphi}_1 = \tilde{\varphi}_2 = 0$$
$$\tilde{\alpha}_2 = \tilde{\beta}_2 = \operatorname{arctg}(N_9 N_8^{-1})^{1/2}, \quad \tilde{\gamma}_2 = \pi/2 + \tilde{\alpha}_2 \tag{3.33}$$
$$N_8 = a - d + c, \quad N_9 = b - d + c$$

It follows from relations (3.33) that the quasi-longitudinal and quasi-transverse waves in the directions $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, which are not the directions of the axes of elastic symmetry of the medium, as also in the directions of the axes of symmetry, change into purely longitudinal and purely transverse waves, since the phase and ray velocity vectors of the quasi-longitudinal and quasi-transverse waves and the displacement vector of the particles of the medium of the quasi-longitudinal wave coincide with the normals to the wave fronts, while the displacement vector of the quasi-transverse wave coincides with the wave front.

From formulae (2.9) and (2.10) and conditions (3.29) and (3.30) we have the following conditions for the directions of the phase and ray velocity vectors:

on the segments $(0, \tilde{\theta}_1)$ and $(0, \tilde{\theta}_2)$

$$0 < \beta_1 < \alpha_1 < \tilde{\alpha}_1, \quad \varphi_1 < 0, \quad 0 < \alpha_2 < \beta_2 < \tilde{\alpha}_2, \quad \varphi_2 > 0 \tag{3.34}$$

on the segments $(\tilde{\theta}_1, a^{-1/2})$ and $(\tilde{\theta}_2, d^{-1/2})$

$$\tilde{\alpha}_1 < \alpha_1 < \beta_1 < \pi/2, \quad \varphi_1 > 0, \quad \tilde{\alpha}_2 < \beta_2 < \alpha_2 < \pi/2, \quad \varphi_2 < 0 \tag{3.35}$$

The phase velocity of the quasi-longitudinal wave in the direction $\alpha_1 = \tilde{\alpha}_1$ has the minimum value, and the phase velocity of the quasi-transverse wave in the direction $\alpha_2 = \tilde{\alpha}_2$ has the maximum value. It follows from conditions (3.34) and (3.35) that the ray velocity vectors (the energy fluxes) deviate from the directions of the normals to the wave fronts in the direction of increasing phase velocities. This property of the energy fluxes explains the reason for the formation of acute-angled edges in the directions $\alpha_2 = \tilde{\alpha}_2$ on the fronts of the quasi-transverse waves from point sources in media for which $N_4 > 0$ and $N_5 > 0$ when the corresponding condition (see [7], condition (2.8), is satisfied, since from the physical point of view the energy flux density vectors determine the directions of propagation of the wave fronts. For example, if at the initial instant of time a quasi-transverse wave with an oval wave front is excited, on parts of the wave front adjacent to the direction $\alpha_2 = \tilde{\alpha}_2$ with maximum phase velocity, the energy flux density vectors deviate from the normals to the wave front in the direction $\alpha_2 = \tilde{\alpha}_2$, forming acute-angled edges [7, Fig. 2].

Case 1(b). If $N_4 > 0$ and $N_5 < 0$ when $N_1 > 0$, then

$$a > b, \quad N_6 > 0, \quad N_7 < 0, \quad K_2 > 0 \tag{3.36}$$

The value of M, according to formulae (1.5), can have different signs.

When $K_2 > 0$ and $M < 0$, according to expressions (3.8) $F_i > 0$ $(i = 1, \dots, 4)$ and F_5 can have different signs.

If $F_5 < 0$, by repeating the discussions carried out for the first combination of conditions (3.15), we obtain

$$Q_{1\theta} < 0, \quad M_{1\theta} < 0, \quad Q_{2\theta} > 0, \quad M_{2\theta} > 0 \tag{3.37}$$

Taking into account the fact that $N_6 > 0$ and $N_7 < 0$, we conclude from relations (3.5), (3.22) and (3.37) that the graphs of the function Q_k and M_k have the form shown in Fig. 3(b). The functions Q_1 and M_1 are negative continuously decreasing functions which satisfy the condition $Q_1 < M_1$. The functions Q_2 and M_2 are positive continuously increasing functions, which satisfy the conditions $Q_2 > M_2$ on the segment $(0, \tilde{\theta}_2)$ and $M_2 > Q_2$ on the segment $(\tilde{\theta}_2, d^{-1/2})$.

When $F_5 > 0$ we arrive at conditions which differ from (3.37) by having the third condition replaced by

$$Q_{2\theta} < 0 \quad (\theta < \theta_{12}^*), \quad Q_{2\theta} > 0 \quad (\theta > \theta_{12}^*) \tag{3.38}$$

Hence it follows that the graphs of Q_k and M_k differ only slightly from the graphs in Fig. 3(b). Only the graph of the function Q_2, which has a positive minimum at the point θ_{12}^* on the segment $(\tilde{\theta}_2, d^{-1/2})$ has a considerable difference; the conditions for the functions Q_2 and M_2 on the segment $(0, d^{-1/2})$ do not change.

When $M > 0$ and $K_2 > 0$, according to expressions (3.8), $F_5 > 0$; it follows from relations (3.14) and (3.36) that $F_i > 0$ ($i = 2, 3, 4$), and F_1 can have different signs. According to expressions (3.9) when $F_1 < 0$ the values of F_i satisfy the first condition of (3.15). In this case the derivatives of the functions Q_k and M_k satisfy conditions (3.18).

Taking inequalities (3.36) into account, we can conclude from relations (3.5), (3.18) and (3.22) that the graphs of the functions Q_2 and M_2 have the form shown in Fig. 3(a), and the functions Q_2 and M_2 satisfy conditions (3.30) in the segment $(0, d^{-1/2})$. The graphs of the functions Q_1 and M_1 have a form similar to the graphs shown in Fig. 3(b), and on the segment $(0, a^{-1/2})$ the following condition is satisfied

$$Q_1 < M_1 < 0 \tag{3.39}$$

It can be shown that we have a similar picture when $F_1 > 0$.

Hence, when $N_1 > 0$, $N_4 > 0$ and $N_5 < 0$, the functions Q_k and M_k satisfy condition (3.39) on the segment $(0, a^{-1/2})$, and satisfy conditions (3.30) on the segment $(0, d^{-1/2})$.

In this case [10] the phase velocities of the quasi-longitudinal waves inside the segment $(0, a^{-1/2})$ have no extremal values, while the phase velocities of the quasi-transverse waves inside the segment $(0, d^{-1/2})$ have a maximum when $\theta = \tilde{\theta}_2$. The phase velocity of the quasi-longitudinal wave has a minimum when $\theta = 0$ in the direction $\alpha = 0$, and a maximum when $\theta = a^{-1/2}$ in the direction $\alpha_1 = \pi/2$ (Fig. 3b).

According to relations (2.9), (2.10), (3.39) and (3.30) the directions of the phase and ray velocity vectors satisfy the conditions

in the segment $(0, a^{-1/2})$

$$0 < \alpha_1 < \beta_1 < \pi/2, \quad \varphi_1 > 0 \tag{3.40}$$

in the segments $(0, \tilde{\theta}_2)$ and $(\tilde{\theta}_2, d^{-1/2})$

$$0 < \alpha_2 < \beta_2 < \tilde{\alpha}_2, \quad \varphi_2 > 0, \quad \tilde{\alpha}_2 < \beta_2 < \alpha_2 < \pi/2, \quad \varphi_2 < 0 \tag{3.41}$$

Hence it follows that the ray velocity vectors are deflected from the normal to the wave fronts towards increasing phase velocities.

Case 1c. If $N_4 < 0$ and $N_5 > 0$ when $N_1 > 0$, we have

$$b > a, \quad N_6 < 0, \quad N_7 > 0, \quad K_2 > 0, \quad M > 0 \tag{3.42}$$

In this case when $N_1 > |N_6|$, it follows from relations (3.11) that $F_i > 0$ ($i = 2, 4, 5$), $F_1 < 0$, and F_3 can have different signs; for values of F_i the first and third set of conditions (3.15) can be satisfied.

When the first set of conditions (3.15) is satisfied, the derivatives of the functions Q_k and M_k satisfy conditions (3.18). Taking inequalities (3.42) into account, we can conclude from relations (3.5), (3.18) and (3.22) that the functions Q_k and M_k have the form shown in Fig. 3(c).

When the third set of conditions of (3.15) is satisfied, the derivatives of the functions Q_k and M_k satisfy conditions (3.27). Taking inequalities (3.42) into account, it follows from relations (3.5), (3.27) and (3.22) that the graphs of functions Q_k and M_k have the form shown in Fig. 3(c). The graph of the function Q_1 has the non-fundamental difference of a maximum of negative sign at the point θ_{11}^* on the segment $(0, a^{-1/2})$.

Consequently, when $N_4 < 0$ and $N_5 > 0$ when $N_1 > 0$ the functions Q_k and M_k on the segment $(0, a^{-1/2})$ satisfy the conditions

$$M_1 < Q_1 < 0 \tag{3.43}$$

and on the segment $(0, d^{-1/2})$ satisfy the conditions (3.30)

In this case [10] the phase velocities of the quasi-longitudinal waves inside the segment $(0, a^{-1/2})$ have no extremal values; the maximum value is reached when $\theta_1 = 0$ in the direction $\alpha_1 = 0$, and the minimum value is reached when $\theta = a^{-1/2}$ in the direction $\alpha_1 = \pi/2$ (Fig. 3c). The phase velocities of the quasi-transverse waves, as in the previous cases, have maximum values when $\theta = \tilde{\theta}_2$ in the direction $\alpha_2 = \tilde{\alpha}_2$, and minimum values when $\theta = 0$ and $\theta = d^{-1/2}$ in the directions $\alpha_2 = 0$ and $\alpha_2 = \pi/2$.

It follows from relations (2.9), (2.10), (3.42) and (3.30) that the directions of the phase and ray velocity vectors of the quasi-longitudinal waves, unlike the previous cases, satisfy the conditions

$$0 < \beta_1 < \alpha_1 < \pi/2, \quad \varphi_1 < 0 \tag{3.44}$$

The directions of the phase and ray velocity vectors of the quasi-transverse waves, as in the previous cases, satisfy conditions (3.41).

Case 2. When $N_1 < 0$, the derivatives in relations (3.12) satisfy conditions (3.13) with opposite signs of the inequalities.
When $N_1 < 0$ three combinations of values of N_4 and N_5 are possible.

Case 2(a). If $N_4 < 0$ and $N_5 < 0$, the following conditions are satisfied

$$N_1 < 0, \quad N_6 < 0, \quad N_7 < 0, \quad K_2 < 0, \quad M < 0 \tag{3.45}$$

From relations (3.8), (3.11) and inequalities (3.45) we have: $F_i < 0$ $(i = 2, 4, 5)$, and F_1 and F_3 can have different signs. When $K_2 < 0$ and $M < 0$, according to relations (3.8) and (3.9), the following combinations of the distribution of the values of F_i are possible

$$\begin{gathered} F_1 > 0, \quad F_5 < F_2 < F_4 < F_3 < 0 \\ F_1 > F_3 > 0, \quad F_5 < F_2 < F_4 < 0 \\ F_5 < F_2 < F_4 < F_3 < F_1 < 0 \end{gathered} \tag{3.46}$$

Repeating the discussions employed when analysing Case 1a, when the first set of conditions (3.46) is satisfied we obtain the following conditions for the derivatives of the functions Q_k and M_k

$$\begin{gathered} Q_{1\theta} > 0, \quad M_{1\theta} < 0, \quad Q_{2\theta} > 0 \ (\theta < \theta_{12}^*), \quad Q_{2\theta} < 0 \\ (\theta > \theta_{12}^*), \quad M_{2\theta} > 0 \ (\theta < \theta_2^*), \quad M_{2\theta} < 0 \ (\theta > \theta_2^*) \end{gathered} \tag{3.47}$$

It follows from relations (3.5) and (3.22) and inequalities (3.47) that when the first set of conditions (3.46) is satisfied the graphs of the functions Q_k and M_k have the form shown in Fig. 4(a). It can be shown that the form of the graphs of the functions Q_k and M_k, when the second and third set of conditions of (3.46) are satisfied, is similar, with the exception of the minimum at the points θ_1^* and θ_{11}^* respectively of the graphs of the function Q_1 in the case of the second set of conditions and of the function M_1 in the case of the third set of conditions (3.46).
Hence, when $N_4 < 0$ and $N_5 < 0$, in the intervals (3.1), the functions Q_k and M_k satisfy the conditions: on the segment $(0, a^{-1/2})$

$$Q_1 < M_1 < 0 \ (\theta < \tilde{\theta}_1), \quad M_1 < Q_1 < 0 \ (\theta > \tilde{\theta}_1) \tag{3.48}$$

on the segment $(0, d^{-1/2})$

$$M_2 > Q_2 > 0 \ (\theta < \tilde{\theta}_2), \quad Q_2 > M_2 > 0 \ (\theta > \tilde{\theta}_2) \tag{3.49}$$

It was established in [9] that when the conditions $N_4 < 0$ and $N_5 < 0$ are satisfied inside the intervals (3.1), the phase velocities of the quasi-longitudinal waves have a maximum when $\theta = \tilde{\theta}_1$, and the quasi-transverse waves have a minimum when $\theta = \tilde{\theta}_2$, given by formulae (3.23). The points of intersection of the graphs of the functions Q_k and M_k (Fig. 4a) correspond to the extreme points.
According to relations (3.31)–(3.33), the quasi-longitudinal and quasi-transverse waves in the directions of the axes of elastic symmetry y and x of the medium and in the directions $\alpha_k = \tilde{\alpha}_k$ with extreme phase velocities, become purely longitudinal and purely transverse waves when $\theta = \tilde{\theta}_k$.
According to formulae (2.9) and (2.10) and conditions (3.48) and (3.49), we have the following conditions for the directions of the phase and beam velocity vectors
on the segment $(0, a^{-1/2})$

$$0 < \alpha_1 < \beta_1, \ < \tilde{\alpha}_1, \quad \varphi_1 > 0 \ (\theta < \tilde{\theta}_1), \quad \tilde{\alpha}_1 < \beta_1 < \alpha_1 < \pi/2, \quad \varphi_1 < 0 \ (\theta > \tilde{\theta}_1) \tag{3.50}$$

on the segment $(0, d^{-1/2})$

$$0 < \beta_2 < \alpha_2 < \tilde{\alpha}_2, \quad \varphi_2 < 0 \ (\theta < \tilde{\theta}_2), \quad \tilde{\alpha}_2 < \alpha_2 < \beta_2 < \pi/2, \quad \varphi_2 > 0 \ (\theta > \tilde{\theta}_2) \tag{3.51}$$

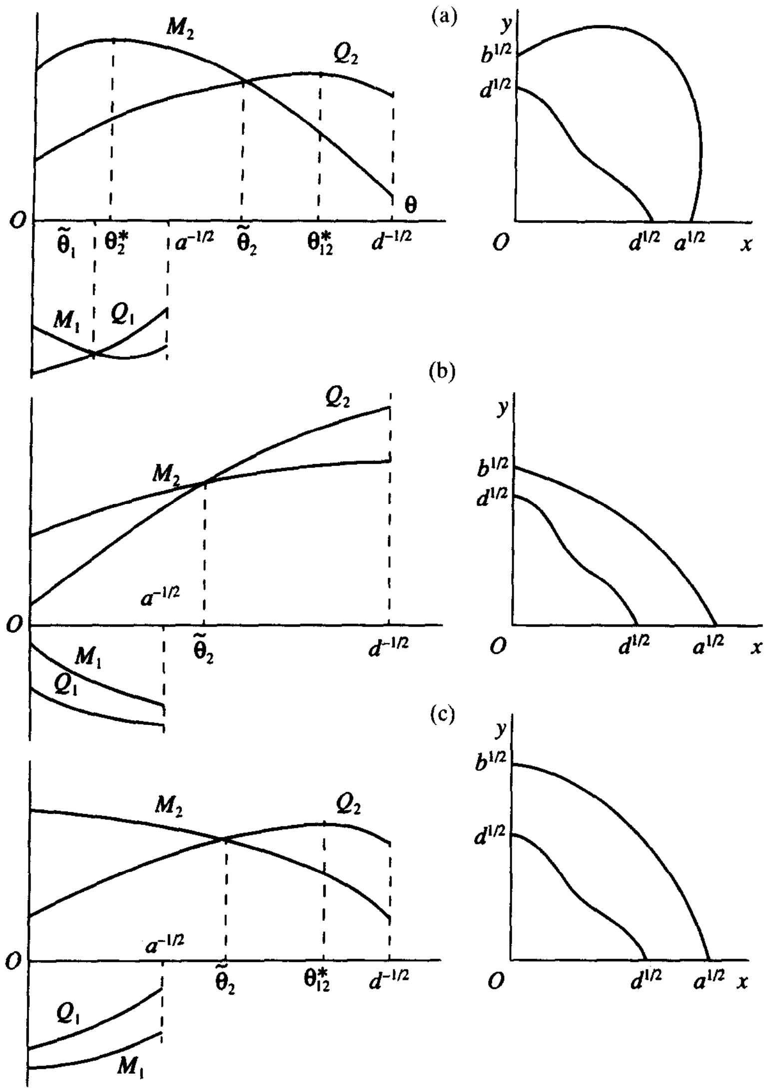

Fig. 4

In follows from conditions (3.50) and (3.51) that here, as in the previous case, the property of the ray velocity vectors (the energy fluxes) of deviating from the directions of the normals to the wave fronts towards an increase in the phase velocities is preserved.

Case 2(b). If $N_4 > 0$ and $N_5 < 0$ when $N_1 < 0$, then

$$a > b, \quad N_6 > 0, \quad N_7 < 0, \quad M < 0 \tag{3.52}$$

and K_2 may have different signs.

In this case, the graphs of the functions Q_k and M_k when $K_2 > 0$ have the form shown in Fig. 4(b), while when $K_2 < 0$ they have a similar form, the graphs of the functions Q_2 and M_2, which have maxima at the points θ_2^* and θ_{12}^*, have only unimportant differences.

Consequently, when $N_4 > 0$ and $N_5 < 0$, the functions Q_k and M_k inside the intervals (3.1) satisfy the following conditions:

on the segment $(0, a^{-1/2})$

$$Q_1 < M_1 < 0 \tag{3.53}$$

on the segment $(0, d^{-1/2})$ they satisfy conditions (3.49).

It was established in [10] that, when the conditions $N_1 < 0$, $N_4 > 0$ and $N_5 < 0$ are satisfied, the phase velocities of the quasi-longitudinal waves on the segment $(0, a^{-1/2})$ increase continuously, and the phase velocities of the quasi-transverse waves have a minimum on the segment $(0, d^{-1/2})$ at the point $\tilde{\theta}_2$ (Fig. 4b).

According to formulae (2.9) and (2.10) and conditions (3.53) and (3.49), the directions of the phase and ray velocity vectors satisfy the conditions

on the segment $(0, a^{-1/2})$

$$0 < \alpha_1 < \beta_1 < \pi/2, \quad \varphi_1 > 0 \tag{3.54}$$

on the segment $(0, d^{-1/2})$ they satisfy conditions (3.51).

Case 2(c). When $N_4 < 0$, $N_5 > 0$ and $N_1 < 0$, the following conditions are satisfied

$$b > a, \quad N_6 < 0, \quad N_7 > 0, \quad K_2 < 0 \tag{3.55}$$

The quantity M can have different signs.

When $M > 0$ the graphs of the functions Q_k and M_k have the form shown in Fig. 4(c), and when $M < 0$ their form changes only slightly: the graph of the function M_1 has a minimum at the point θ_1^*.

Consequently, when $N_4 < 0$, $N_5 > 0$ and $N_1 < 0$, the functions Q_k and M_k on sections (3.1) satisfy the following conditions:

on the segment $(0, a^{-1/2})$

$$M_1 < Q_1 < 0 \tag{3.56}$$

on the segment $(0, d^{-1/2})$ they satisfy conditions (3.49).

In this case, according to [10], the phase velocities of the quasi-longitudinal waves on the segment $(0, a^{-1/2})$ decrease continuously, while the phase velocities of the quasi-transverse waves on the segment $(0, d^{-1/2})$ have a minimum at the point $\tilde{\theta}_2$ (Fig. 4c).

It follows from formulae (2.9) and (2.10) and conditions (3.56) and (3.49) that the directions of the phase and ray velocity vectors satisfy the following conditions:

on the segment $(0, a^{-1/2})$

$$0 < \beta_1 < \alpha_1 < \pi/2, \quad \varphi_1 < 0 \tag{3.57}$$

on the segment $(0, d^{-1/2})$ they satisfy conditions (3.51).

Hence, the results of an analysis show that when the conditions $N_2 > 0$ and $N_3 > 0$ are satisfied in all the cases considered above, the functions Q_k and M_k inside the intervals (3.1) satisfy the inequalities

$$Q_1 < 0, \quad M_1 < 0, \quad Q_2 > 0, \quad M_2 > 0 \tag{3.58}$$

It follows from relations (2.4), (3.4) and (3.58) that

$$S_{x1} > 0, \quad S_{y1} < 0, \quad S_{x2} > 0, \quad S_{y2} < 0 \tag{3.59}$$

Consequently, when $N_2 > 0$ and $N_3 > 0$ the projections of the energy flux density vectors (2.4) of the quasi-longitudinal waves ($k = 1$) and quasi-transverse waves ($k = 2$) (1.3), determined in sections (3.1), have the following directions: S_{x1} and S_{x2} are directed along the x axis, and S_{y1} and S_{y2} are directed along the negative y semi-axis.

4. ANALYSIS OF THE SOLUTIONS WHEN $N_2 > 0$ AND $N_3 < 0$

When $N_2 > 0$ the solutions are determined on the Riemann surface shown in Fig. 1. When $N_2 > 0$ and $N_3 < 0$ the following conditions are satisfied

$$a > b, \quad N_1 < 0, \quad N_7 < 0, \quad K_2 < 0, \quad M < 0 \tag{4.1}$$

The quantity N_6 can have different signs.

The Case $N_6 < 0$. When $a > b$ we have

$$|N_6| < |N_1| < |N_7|, \quad (c^2 - d^2) > ad > bd \tag{4.2}$$

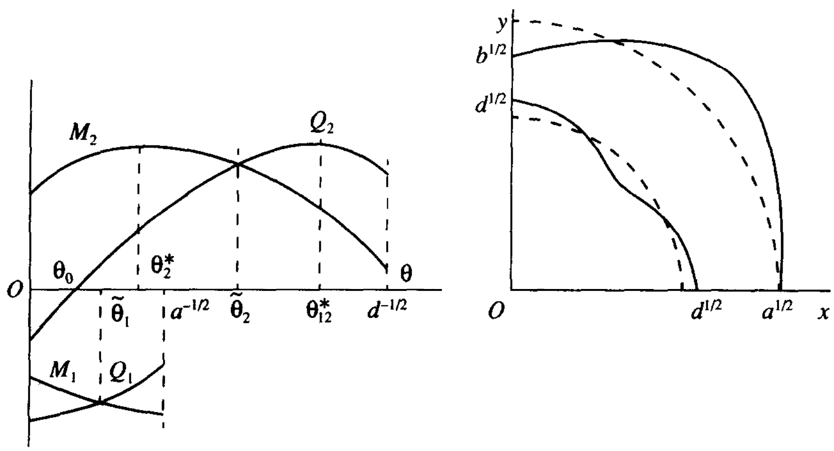

Fig. 5

We conclude from relations (3.11), (4.1) and (4.2) that $F_1 > 0$, $F_i < 0$ ($i = 2, 4, 5$), and F_3 can have different signs. According to relations (3.8) and (3.9) when $F_3 < 0$ the values of F_i satisfy the first set of conditions (3.46) and when $F_3 > 0$ the values of F_i satisfy the second set of conditions (3.46). When the first set of conditions (3.46) is satisfied, the derivatives of the functions Q_k and M_k satisfy conditions (3.47).

Taking into account the fact that $N_6 < 0$, $N_7 < 0$ and $N_3 < 0$, we conclude from relations (3.5), (3.22) and (3.47) that the graphs of the functions Q_k and M_k have the form shown in Fig. 5. It can be shown that when the second set of conditions (3.46) is satisfied the graphs of the functions Q_k and M_k have a similar form, and only the graph of the function Q_1, which has a minimum at the point θ_{11}^*, is different.

Consequently, the functions Q_k and M_k on sections (3.1) satisfy the following conditions

on the segment $(0, a^{-1/2})$ – conditions (3.48),

on the segment $(0, d^{-1/2})$

$$Q_2 < 0, \quad M_2 > 0 \ (\theta < \theta_0), \quad M_2 > Q_2 > 0 \ (\theta_0 < \theta < \tilde{\theta}_2), \quad Q_2 > M_2 > 0 \ (\theta > \tilde{\theta}_2) \tag{4.3}$$

It follows from relations (2.9), (2.10) and (2.3) and (4.3) that the directions of the phase and ray velocity vectors of the quasi-longitudinal waves satisfy conditions (3.50), and the quasi-transverse waves satisfy the following conditions

on the segment $(0, \theta_0)$

$$\alpha_2 > 0, \quad \beta_2 < 0, \quad \varphi_2 = -(\alpha_2 + |\beta_2|) < 0 \tag{4.4}$$

on the segment $(\theta_0, d^{-1/2})$ they satisfy conditions (3.51).

At the boundaries of the segment $(0, \theta_0)$, by formulae (2.9) and (2.10) when $k = 2$, the angles which determine the directions of the phase and ray velocity vectors of the quasi-transverse waves have the following values: $\alpha_2 = \beta_2 = \varphi_2 = 0$ when $\theta = 0$ and $\alpha_2 > 0$, $\beta = 0$ and $\varphi_2 = -\alpha_2$ when $\theta = \theta_0$.

Consequently, when $\theta = 0$ and $\theta = \theta_0$ the ray velocity vectors of the quasi-transverse waves are directed along the negative y semi-axis.

It follows from conditions (4.4) that the quasi-transverse waves, defined on the segment $(0, \theta_0)$, unlike the previous cases, propagate for positive angles α_2 of the phase velocity vectors with negative values of the angles θ_2 of the ray velocity vectors. This feature has a direct connection with the existence on the wave fronts of the quasi-transverse waves of point sources of acute-angle edges, which propagate in the direction of the y axis when the conditions $N_3 < 0$ are satisfied for the constants of elasticity [7, 8], and is the reason for the formation of acute-angle edges.

The case $N_6 > 0$. Repeating the discussion carried out when analysing Case 2b, and taking into account the inequality $N_3 < 0$, it can be shown that the graphs of the functions Q_1 and M_1 have the form shown in Fig. 4(b), while the graphs of the functions Q_2 and M_2 have the form shown in Fig. 5. Consequently, in the intervals (3.1) the functions Q_1 and M_1 satisfy conditions (3.53), while the functions Q_2 and M_2 satisfy conditions (4.3). The directions of the phase and ray velocity vectors of the quasi-longitudinal waves satisfy conditions (3.54), while the quasi-transverse waves on the segment $(0, \theta_0)$ satisfy conditions

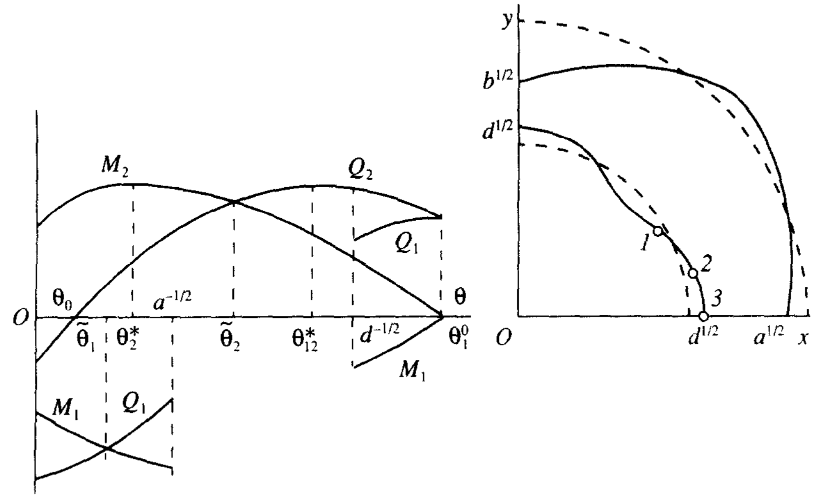

Fig. 6

(4.4), on the segment $(\theta_0, \tilde{\theta}_2)$ they satisfy the first set of conditions (3.51), and on the segment $(\tilde{\theta}_2, d^{-1/2})$ they satisfy the second set of conditions (3.51).

It follows from relations (2.4), (3.4), (3.48), (3.53) and (4.3) that when $N_2 > 0$ and $N_3 < 0$ in the intervals (3.1), the projections of the energy flux density vectors onto the coordinate axes satisfy the conditions

$$S_{x1} > 0, \quad S_{y1} < 0, \quad S_{x2} < 0 \ (\theta < \theta_0), \quad S_{x2} > 0 \ (\theta > \theta_0), \quad S_{y2} < 0 \tag{4.5}$$

Hence it follows that, unlike conditions (3.59), when $N_2 > 0$ and $N_3 > 0$, the projections S_{x2} of he energy flux density vectors of the quasi-transverse waves, defined on the segment $(0, \theta_0)$, are directed along the negative x semi-axes. When $\theta = 0$ and $\theta = \theta_0$ the projections $S_{x2} = 0$ of the energy flux density vector are directed along the negative y semi-axis.

5. ANALYSIS OF THE SOLUTIONS WHEN $N_2 < 0$ AND $N_3 < 0$

When $N_2 < 0$ the solutions are defined on the Riemann surface, shown in Fig. 2. The waves (1.3), propagating in the directions $0 \leqslant \alpha_k \leqslant \pi/2$, are defined on the following parts of the Riemann surface [6]: the quasi-longitudinal waves ($k = 1$) on the segment $(0, a^{-1/2})$ of the upper edge of the cut $(-a^{-1/2}, +a^{-1/2})$ of the θ_1 plane, and the quasi-transverse waves when ($k = 2$) on the segment $(0, \theta_1^0)$ of the upper edge of the cut $(-\theta_1^0, +\theta_1^0)$ of the θ_2 plane, and when ($k = 1$) on the lower edge of the cut $(+d^{-1/2}, +\theta_1^0)$ of the θ_1 plane.

Since the conditions $N_4 < 0$ and $N_5 < 0$ are satisfied when $N_2 < 0$ and $N_3 < 0$, according to the results obtained earlier in [10, 11] the graphs of the phase velocities have the form shown in Fig. 6. On the graph of the phase velocities of the quasi-transverse waves the points indicate value of the velocities corresponding to the boundaries of the segments $(d^{-1/2}, \theta_1^0)$ in the θ_1 and θ_2 planes of the Riemann surface (Fig. 2): 1 – the point $\theta = d^{-1/2}$ in the θ_2 plane, 2 – the branch point θ_1^0, and 3 – the point $\theta = d^{-1/2}$ in the θ_1 plane.

According to relations (1.5) and (3.45) the positive real branch point of the inner radical of functions (1.4) corresponds to the value

$$\theta_1^0 = \{[M - (-4bdc^2N_1)^{1/2}](K_1K_2)^{-1}\}^{1/2} \tag{5.1}$$

On the upper edge of the cut $(-\theta_1^0, +\theta_1^0)$ of the θ_2 plane (Fig. 2) the function A_2 and B_2 are given by expressions (3.6) when $k = 2$. When going round the branch point θ_1^0 in a clockwise direction from the lower edge of the cut $(-\theta_1^0, +\theta_1^0)$ of the θ_2 plane on the upper edge of the cut $(+d^{-1/2}, +\theta_1^0)$ of the θ_1 plane, the functions A_2 and B_2 take values of A_1 and B_1, given by expressions (3.6) with $k = 1$. Similarly, the functions Q_2 and M_2, defined by expressions (2.5) when $k = 2$, on changing from the lower edge $(-\theta_1^0, +\theta_1^0)$ of the θ_2 plane to the upper edge of the cut $(d^{-1/2}, +\theta_1^0)$ of the θ_1 plane, take Q_1 and M_1, defined by expressions (2.5) with $k = 1$.

Repeating the discussions used when analysing Case 1, and taking into account the fact that when $N_2 < 0$ and $N_3 < 0$, conditions (3.45) are satisfied, it can be shown that the graphs of the functions Q_k and M_k have the form shown in Fig. 6.

Consequently, the functions Q_k and M_k on the segments $(0, a^{-1/2})$ and $(0, \theta_1^0)$ of the upper edges of the cuts $(-a^{-1/2}, +a^{-1/2})$ and $(-\theta_1^0, +\theta_1^0)$ of the θ_1 and θ_2 planes and on the lower edge of the cut $(+d^{-1/2}, +\theta_1^0)$ of the θ_1 plane satisfy the following conditions:

on the segment $(0, a^{-1/2})$ – conditions (3.48),

on the segment $(0, \theta_1^0)$ – conditions (4.3), and

on the segment $(d^{-1/2}, \theta_1^0)$

$$Q_1 > 0, \quad M_1 < 0 \tag{5.2}$$

At the point $\tilde{\theta}_2$ we have $Q_2 = M_2$, and at the branch point θ_1^0 we have $Q_2 = Q_1$ and $M_2 = M_1 = 0$.

According to formulae (2.9) and (2.10) and relations (3.48), (4.3) and (5.2) the directions of the phase and ray velocity vectors of the quasi-longitudinal waves ($k = 1$), defined on the segment $(0, a^{-1/2})$ of the θ_1 plane, and of the quasi-transverse waves, defined with $k = 2$ on the segment $(0, \theta_1^0)$ of the θ_2 plane, and with $k = 1$ on the segment $(d^{-1/2}, \theta_1^0)$ of the θ_1 plane, satisfy the following conditions

on the segment $(0, a^{-1/2})$ – conditions (3.50),

on the segment $(0, \theta_0)$ – conditions (4.4),

on the segment (θ_0, θ_1^0) – conditions (3.51), and

on the segment $(d^{-1/2}, \theta_1^0)$

$$\beta_1 > \pi/2, \quad \pi/2 > \alpha_1 > \alpha_1(\theta_1^0) = \alpha_2(\theta_1^0), \quad \varphi_1 > 0 \tag{5.3}$$

According to relations (4.4) the quasi-transverse waves ($k = 2$), defined on the segment $(0, \theta_0)$, propagate with positive angles α_2 of the phase velocity vectors for negative values of the angles β_2 of the ray velocity vectors, where the angles $\beta_2 = 0$ at the boundaries of the segment $(0, \theta_0)$, i.e. the directions of the ray velocity vectors coincide with the direction of the negative y semi-axis. This explains the reason for the formation on the wave fronts of the quasi-transverse waves of point sources of acute-angled edges, which propagate in the direction of the axis of symmetry y when $N_3 < 0$ [7, 8].

It follows from relations (5.3) that the directions of the phase and ray velocity vectors of the quasi-transverse waves (1.3), defined with $k = 1$ on the segment $(d^{-1/2}, \theta_1^0)$ of the θ_1 plane of the Riemann surface, satisfy the conditions $\alpha_1 < \pi/2$ and $\beta_1 > \pi/2$. On the boundaries of the segment $(d^{-1/2}, \theta_1^0)$ the angles β_1 have the same values of $\pi/2$, and consequently, at these points the directions of the ray velocity vectors of the quasi-transverse waves coincide with the direction of the positive x semi-axis. This explains the reason for the formation on the wave fronts of the quasi-transverse waves of point sources of acute-angled edges, propagating in the direction of the x axis of symmetry when $N_2 < 0$ [7, 8].

When $N_2 < 0$ and $N_3 < 0$, as in the previous cases, in the directions of the axes of elastic symmetry of the medium and in the directions α_1 and α_2 with extreme phase velocities the quasi-longitudinal and quasi-transverse waves become purely longitudinal and purely transverse waves.

Since the conditions $p_2 > 0$ and $p_1 > 0$ are satisfied on the segments $(0, \theta_1^0)$ of the θ_2 plane and $(d^{-1/2}, \theta_1^0)$ of the θ_1 plane respectively, then by relations (2.4), (4.3) and (5.2) the projections of the energy flux density vectors of the quasi-transverse waves (1.3) with $k = 2$ and $k = 1$ onto the coordinate axes satisfy the following conditions:

on the segment $(0, \theta_1^0)$

$$S_{x2} < 0, \quad S_{y2} < 0 \quad (\theta < \theta_0), \quad S_{x2} > 0, \quad S_{y2} < 0 \quad (\theta > \theta_0) \tag{5.4}$$

on the segment $(d^{-1/2}, \theta_1^0)$

$$S_{x1} > 0, \quad S_{y1} > 0 \tag{5.5}$$

Consequently, when $N_2 < 0$ and $N_3 < 0$, as in the previous case ($N_2 > 0$, $N_3 < 0$), the directions of the projections S_{xs} and S_{y2} of the energy flux density vectors of the quasi-transverse waves ($k = 2$), defined on the segment $(0, \theta_0)$ of the θ_2 plane, coincide with the directions of the negative x and y semi-axes. The directions of the projections S_{x1} and S_{y1} of the energy flux density vectors of the quasi-transverse waves ($k = 1$), defined on the segment $(d^{-1/2}, \theta_1^0)$ of the θ_1 plane, unlike all the cases previously considered, coincide with the directions of the positive x and y semi-axes.

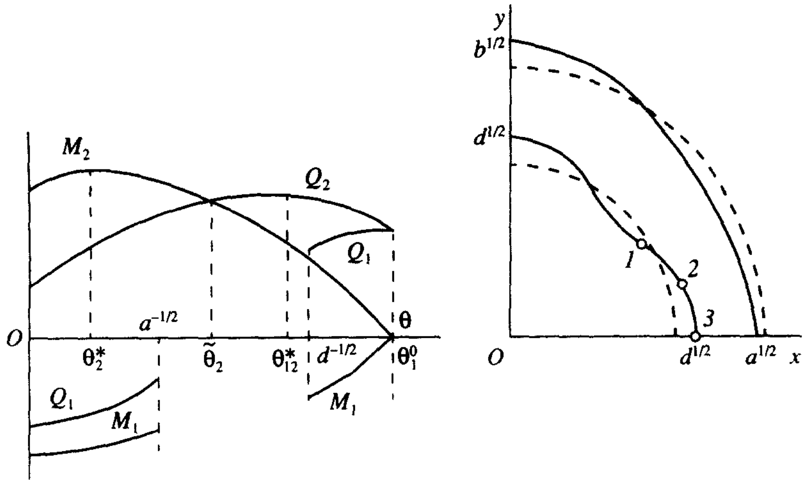

Fig. 7

6. ANALYSIS OF THE SOLUTIONS WHEN $N_2 < 0$ AND $N_3 > 0$

In this case, unlike the preceding case, the following conditions are satisfied

$$b > a, \quad N_1 < 0, \quad N_6 < 0, \quad M < 0, \quad K_2 < 0 \tag{6.1}$$

The quantity N_7 can have different signs.

If $N_7 > 0$, then when $N_2 < 0$ and $N_3 > 0$, according to conditions (6.1), the conditions $N_1 < 0$, $N_4 < 0$ and $N_5 > 0$ are satisfied. In this case the graphs of the phase velocities and of the functions Q_k and M_k have the form shown in Fig. 7.

By formulae (2.9) and (2.10) and the graphs of the functions Q_k and M_k the directions of the phase and ray velocity vectors of the quasi-longitudinal and quasi-transverse waves satisfy conditions (3.57) on the segment $(0, a^{-1/2})$, satisfy conditions (3.51) on the segment $(0, \theta_1^0)$, and satisfy conditions (5.3) on the segment $(d^{-1/2}, \theta_1^0)$.

When $N_7 < 0$ the conditions $N_4 < 0$ and $N_5 < 0$ are satisfied. In this case the graphs of the functions Q_1 and M_1 and of the phase velocities of the quasi-longitudinal waves, defined on the segment $(0, a^{-1/2})$, have the form shown in Fig. 6. The graphs of the functions Q_2 and M_2, defined on the segment $(0, \theta_1^0)$, and of the function Q_1 and M_1, defined on the segment $(d^{-1/2}, \theta_1^0)$, and the phase velocities of the quasi-transverse waves have the form shown in Fig. 7.

Consequently, when $N_7 < 0$, the directions of the phase velocity and ray velocity vectors of the quasi-longitudinal waves, defined on the segment $(0, a^{-1/2})$, and of the quasi-transverse waves, defined on the segments $(0, \theta_1^0)$ and $(d^{-1/2}, \theta_1^0)$, satisfy conditions (3.50) on the segment $(0, a^{-1/2})$, satisfy conditions (3.51) on the segment $(0, \theta_1^0)$, and satisfy condition (5.3) on the segment $(d^{-1/2}, \theta_1^0)$.

When $N_2 < 0$ and $N_3 > 0$, the projections of the energy flux density vectors of the quasi-longitudinal waves onto the coordinate axes satisfy conditions (4.8); for the quasi-transverse waves they satisfy the conditions $S_{x2} > 0$ and $S_{y2} < 0$ on the segment $(0, \theta_1^0)$ of the θ_2 plane, and they satisfy conditions (5.5) on the segment $(d^{-1/2}, \theta_1^0)$ of the θ_1 plane.

Hence, we have obtained a complete solution of the problem of investigating the behaviour of the propagation of the energy of elastic waves as a function of the directions of motion of the waves and the ratios of the constants of elasticity of the media for all practical anisotropic media of the class considered.

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